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FRACTIONAL SOLUTIONS OF A CONFLUENT HYPERGEOMETRIC EQUATION

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Abstract. By means of fractional calculus techniques, we find explicit solutions of confluent hypergeometric equations. We use the $N-$fractional calculus operator N^{μ} method to derive the solutions of these equations.

1. Introduction and preliminaries

Fractional calculus is "the theory of derivatives and integrals of any arbitrary real or complex order, which unify and generalize the notions of integer-order differentiation and $n-$fold integration" [3, 8]. The idea of generalizing differential operators to a non-integer order, in particular to the order 1/2, first appears in the correspondence of Leibniz with L'Hôpital (1695), Johann Bernoulli (1695), and John Wallis (1697) as a mere question or maybe even play of thoughts. In the following three hundred years a lot of mathematicians contribute to the fractional calculus: L. Euler, J. L. Lagrange, P. S. Laplace, S. F. Lacroix, J. B. J. Fourier, N. H. Abel, J. Liouville, S. S. Greatheed, A. De Morgan, B. Riemann, W. Center, H. Holmgren, A. K. Grünwald, A.V. Letnikov, H. Laurent, O. Heaviside, G. H. Hardy, H. Weyl, E. L. Post, H. T. Davis, A. Erdélyi, H. Kober, A. Zygmund, M. Riesz, I.M. Gel'fand, G. E. Shilov, I. N. Sneddon, S. G. Samko, T. J. Osler, E. R. Love, and many others [7, 11].

The differintegration operators and their generalizations [5, 6, 9, 10] have been used to solve some classes of differential equations and fractional differential equations. Furthermore we can note that the fractional differential equations are playing an important role in fluid dynamics,

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traffic model with fractional derivative, measurement of viscoelastic material properties, modeling of viskoplasticity, control theory, economy, nuclear magnetic resonance, geometric mechanics, mechanics, optics, signal processing and so on.

Two of the most commonly encountered tools in the theory and applications of fractional calculus are provided by the Riemann-Liouville operatorR_z^{υ} $(\upsilon \in \mathbb{C})$ and the Weyl operator W_z^{υ} $(\upsilon \in \mathbb{C})$, which are defined by [1, 2, 8, 11]

$$R_z^{\upsilon} f(z) = \begin{cases} \frac{1}{\Gamma(\upsilon)} \int_0^z (z-t)^{\upsilon-1} f(t)dt : \ Re(\upsilon) > 0, \\ \frac{d^n}{dz^n} R_z^{\upsilon+n} f(z) : \ -n < Re(\upsilon) \leq 0; \ n \in \mathbb{N}, \end{cases} \tag{1.1}$$

and

$$W_z^{\upsilon} f(z) = \begin{cases} \frac{1}{\Gamma(\upsilon)} \int_z^{\infty} (t-z)^{\upsilon-1} f(t)dt : \ Re(\upsilon) > 0, \\ \frac{d^n}{dz^n} W_z^{\upsilon+n} f(z) : \quad -n < Re(\upsilon) \leq 0; \ n \in \mathbb{N}, \end{cases} \tag{1.2}$$

provided that the defining integrals in (1.2) and (1.3) exist, $\mathbb{N}$ being the set of positive integers.

Definition 1.1. (cf. [4, 5, 12]) Let

$$D = \left\{D^-, D^+\right\}, \quad C = \left\{C^-, C^+\right\},$$

C^- be a curve along the cut joing two points z and $-\infty + iIm(z)$,
C^+ be a curve along the cut joing two points z and $\infty + iIm(z)$,
D^- be a domain surrounded by C^-, D^+ be a domain surrounded by C^+. (Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in D $(z \in D)$,

$$\begin{aligned} f_\mu(z) &= (f(z))_\mu \\ &= \frac{\Gamma(\mu+1)}{2\pi i} \int_C \frac{f(t)\,dt}{(t-z)^{\mu+1}} \quad \left(\mu \in \mathbb{R} \setminus \mathbb{Z}^-; \mathbb{Z}^- = \{-1, -2, ...\}\right) \end{aligned}$$

and

$$f_{-n}(z) = \lim_{\mu \to -n} f_\mu(z) \quad \left(n \in \mathbb{Z}^+\right),$$

where $t \neq z$,

$$-\pi \leq \arg(t-z) \leq \pi \text{ for } C^-$$

and

$$0 \leq \arg(t-z) \leq 2\pi \text{ for } C^+,$$

then $f_\mu(z)$ $(\mu > 0)$ is said to be the fractional derivative of $f(z)$ of order μ and $f_\mu(z)$ $(\mu < 0)$ is said to be the fractional integral of (z) of order $-\mu$, provided (in each case) that $|f_\mu(z)| < \infty$ $(\mu \in \mathbb{R})$.

Finally, let the fractional calculus operator (Nishimoto's operator) N^μ be defined by (cf.[5])

$$N^\mu = \left(\frac{\Gamma(\mu+1)}{2\pi i} \int_C \frac{dt}{(t-z)^{\mu+1}} \right) \quad (\mu \notin \mathbb{Z}^-)$$

with

$$N^{-n} = \lim_{\mu \to -n} N^\mu \quad \left(n \in \mathbb{Z}^+\right).$$

We find it to be worthwhile to recall here the following useful lemmas and properties associated with the fractional differintegration which is defined above (cf.e.g.[4, 5]).

Lemma 1.2. *(Linearity property). If the functions $f(z)$ and $g(z)$ are single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$(h_1 f(z) + h_2 g(z))_\mu = h_1 f_\mu(z) + h_2 g_\mu(z) \quad (\mu \in \mathbb{R}; z \in \Omega) \tag{1.3}$$

for any constants h_1 and h_2.

Lemma 1.3. *(Index law). If the function $f(z)$ is single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$(f_\gamma(z))_\mu = f_{\gamma+\mu}(z) = (f_\mu(z))_\gamma \quad (f_\gamma(z) \neq 0; f_\mu(z) \neq 0; \gamma, \mu \in \mathbb{R}; z \in \Omega). \tag{1.4}$$

Lemma 1.4. *(Generalized Leibniz rule). If the functions $f(z)$ and $g(z)$ are single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$(f(z).g(z))_\mu = \sum_{n=0}^{\infty} \binom{\mu}{n} f_{\mu-n}(z).g_n(z) \quad (\mu \in \mathbb{R}; z \in \Omega), \tag{1.5}$$

where $g_n(z)$ is the ordinary derivative of $g(z)$ of order n $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$, it being tacitly assumed (for simplicity) that $g(z)$ is the polynomial part (if any) of the product $f(z)g(z)$.

Remark 1.5. For a constant λ,

$$\left(e^{\lambda z}\right)_\mu = \lambda^\mu e^{\lambda z} \quad (\lambda \neq 0; \mu \in \mathbb{R}; z \in \mathbb{C}). \tag{1.6}$$

Remark 1.6. For a constant λ,

$$\left(e^{-\lambda z}\right)_\mu = e^{-i\pi\mu} \lambda^\mu e^{-\lambda z} \quad (\lambda \neq 0; \mu \in \mathbb{R}; z \in \mathbb{C}). \tag{1.7}$$

Remark 1.7. For a constant λ,

$$\left(z^\lambda\right)_\mu = e^{-i\pi\mu} \frac{\Gamma(\mu-\lambda)}{\Gamma(-\lambda)} z^{\lambda-\mu}, \left(\mu \in \mathbb{R}; z \in \mathbb{C}; \left|\frac{\Gamma(\mu-\lambda)}{\Gamma(-\lambda)}\right| < \infty\right). \tag{1.8}$$

2. N^{μ} method applied to a confluent hypergeometric equation

Theorem 2.1. *Let* $y\epsilon\{y : 0 \neq |y_\mu| < \infty; \mu\epsilon\mathbb{R}\}$ *and* $f\epsilon\{f : 0 \neq |f_\mu| < \infty; \mu\epsilon\mathbb{R}\}$. *Then the nonhomogeneous confluent hypergeometric equation*

$$L[y, x; \gamma, \tau] = y_2 x + y_1 (\gamma - x) - y\tau = f \quad (x \neq 0) \tag{2.1}$$

has particular solutions of the forms:

$$y = \left[\left(f_{-\tau} e^{-x} x^{\gamma-\tau-1}\right)_{-1} e^x x^{\tau-\gamma}\right]_{\tau-1} \equiv y_{(I)}, \tag{2.2}$$

$$y = x^{1-\gamma} \left\{\left[\left(f x^{\gamma-1}\right)_{\gamma-\tau-1} e^{-x} x^{-\tau}\right]_{-1} e^x x^{\tau-1}\right\}_{\tau-\gamma} \equiv y_{(II)}. \tag{2.3}$$

Where $y_n = \frac{d^n y}{dx^n}$ $(n = 0, 1, 2)$, $y_0 = y = y(x)$, $x \in \mathbb{C}$, γ *and* τ *are given constants.*

Proof.

$(proof - (2.2))$. Operate $N-$fractional calculus operator N^{μ} directly to the both sides of (2.1), we then obtain

$$(y_2 x)_\mu + [y_1 (\gamma - x)]_\mu - (y\tau)_\mu = (f)_\mu. \tag{2.4}$$

Using $(1.3), (1.4), (1.5)$ we have

$$(y_2 x)_\mu = y_{2+\mu} x + y_{1+\mu}\mu \tag{2.5}$$

and

$$[y_1 (\gamma - x)]_\mu = y_{1+\mu} (\gamma - x) - y_\mu \mu. \tag{2.6}$$

Making use of the relations (2.5) and (2.6), we may write (2.4) in the following form:

$$y_{2+\mu} x + y_{1+\mu} (\mu + \gamma - x) - y_\mu (\mu + \tau) = f_\mu. \tag{2.7}$$

Chose μ such that

$$\mu = -\tau, \tag{2.8}$$

we have then

$$y_{2-\tau} x + y_{1-\tau} (-\tau + \gamma - x) = f_{-\tau} \tag{2.9}$$

from (2.7).

Therefore, setting

$$y_{1-\tau} = u = u(x) \quad (y = u_{\tau-1}), \tag{2.10}$$

we have

$$u_1 + u\left[(\gamma - \tau)\, x^{-1} - 1\right] = f_{-\tau} x^{-1} \tag{2.11}$$

from (2.9). This is an ordinary differential equation of the first order which has a particular solution:

$$u = \left(f_{-\tau} e^{-x} x^{\gamma-\tau-1}\right)_{-1} e^x x^{\tau-\gamma}. \tag{2.12}$$

Thus we obtain the solution (2.2) from (2.10) and (2.12).

Inversely, the function given by (2.12) satisfies (2.11) clearly. Hence (2.2) satisfies equation (2.9). Therefore, the function (2.2) satisfies equation (2.1).

$(proof - (2.3))$. Set

$$y = x^\alpha \phi, \quad \phi = \phi(x), \tag{2.13}$$

hence

$$y_1 = \alpha x^{\alpha-1}\phi + x^\alpha \phi_1 \tag{2.14}$$

and

$$y_2 = \alpha(\alpha-1)\, x^{\alpha-2}\phi + 2\alpha x^{\alpha-1}\phi_1 + x^\alpha \phi_2. \tag{2.15}$$

Substitute (2.13), (2.14) and (2.15) into (2.1), we have

$$\phi_2 x + \phi_1(2\alpha + \gamma - x) + \phi\left[\alpha(\alpha+\gamma-1)\, x^{-1} - \alpha - \tau\right] = f x^{-\alpha}. \tag{2.16}$$

Here, we choose α such that

$$\alpha(\alpha + \gamma - 1) = 0,$$

that is,

$$\alpha_1 = 0, \quad \alpha_2 = 1 - \gamma.$$

In the case $\alpha = 0$, we have the same results as $(1 - i)$.

Let $\alpha = 1 - \gamma$. From (2.13) and (2.16) we have

$$y = x^{1-\gamma}\phi \tag{2.17}$$

and

$$\phi_2 x + \phi_1(2 - \gamma - x) + \phi(\gamma - \tau - 1) = f x^{\gamma-1}, \tag{2.18}$$

respectively.

Applying the operator N^μ to both members of (2.18), we have

$$\phi_{2+\mu} x + \phi_{1+\mu}(\mu - \gamma - x + 2) + \phi_\mu(-\mu + \gamma - \tau - 1) = \left(f x^{\gamma-1}\right)_\mu. \tag{2.19}$$

Here we choose μ such that

$$-\mu + \gamma - \tau - 1 = 0$$

that is

$$\mu = \gamma - \tau - 1. \tag{2.20}$$

Substituting (2.20) into (2.19) , we have

$$\phi_{1+\gamma-\tau} + \phi_{\gamma-\tau}\left[(-\tau+1)\,x^{-1} - 1\right] = \left(fx^{\gamma-1}\right)_{\gamma-\tau-1} x^{-1}. \tag{2.21}$$

Set

$$\phi_{\gamma-\tau} = w = w\,(x) \quad (\phi = w_{\tau-\gamma})\,, \tag{2.22}$$

we have then

$$w_1 + w\left[(-\tau+1)\,x^{-1} - 1\right] = \left(fx^{\gamma-1}\right)_{\gamma-\tau-1} x^{-1} \tag{2.23}$$

from (2.21) . A particular solution of ordinary differential equation (2.23) is given by

$$w = \left[\left(fx^{\gamma-1}\right)_{\gamma-\tau-1} e^{-x}x^{-\tau}\right]_{-1} e^{x}x^{\tau-1}. \tag{2.24}$$

Therefore, we obtain the solution (2.3) from (2.17) , (2.22) and (2.24) .
Inversely, (2.24) satisfies (2.23) , then

$$\phi = w_{\tau-\gamma},$$

satisfies (2.21) . Therefore (2.3) satisfies (2.1) , since we have (2.13) . □

Theorem 2.2. *If* $y \in \overset{o}{\wp}$, *just as in Theorem 2.1, then the homogeneous confluent hypergeometric equation*

$$L\,[y, x; \gamma, \tau] = y_2 x + y_1\,(\gamma - x) - y\tau = 0 \quad (x \neq 0) \tag{2.25}$$

has solutions of the forms:

$$y = \ell\left(e^{x}x^{\tau-\gamma}\right)_{\tau-1} \equiv y^{I}, \tag{2.26}$$

$$y = \ell x^{1-\gamma}\left(e^{x}x^{\tau-1}\right)_{\tau-\gamma} \equiv y^{II}, \tag{2.27}$$

where ℓ *is an arbitrary constant.*

Proof. When $f = 0$ in Theorem 2.1, we have

$$u_1 + u\left[(\gamma-\tau)\,x^{-1} - 1\right] = 0 \tag{2.28}$$

and

$$w_1 + w\left[(-\tau+1)\,x^{-1} - 1\right] = 0 \tag{2.29}$$

instead of (2.11) and (2.23) , respectively.
Therefore, we obtain (2.26) for (2.28) and (2.27) for (2.29) . □

3. Some applications

EXAMPLE 3.1. *If we substitute $\tau = \frac{5}{2}$ and $\gamma = 2$ in equation (2.1), then we obtain the following equation,*

$$xy_2 + (2 - x)\, y_1 - \frac{5}{2} y = x. \tag{3.1}$$

By using Theorem 2.1, its solutions are

$$y' = \left[\left(x_{-\frac{5}{2}} e^{-x} x^{-\frac{3}{2}} \right)_{-1} e^x x^{\frac{1}{2}} \right]_{\frac{3}{2}} \tag{3.2}$$

and

$$y'' = x^{-1} \left\{ \left[\left[x^2 \right]_{-\frac{3}{2}} e^{-x} x^{-\frac{5}{2}} \right]_{-1} e^x x^{\frac{3}{2}} \right\}_{\frac{1}{2}} \tag{3.3}$$

Actually, if we perform necassary operations in (3.2), we get

$$y' = \left[\left(\frac{16}{105\sqrt{\pi}} x^2 e^{-x} \right)_{-1} e^x x^{\frac{1}{2}} \right]_{\frac{3}{2}}, \tag{3.4}$$

where Riemann Liouville operator is

$$[x]_{-\frac{5}{2}} = \frac{1}{\Gamma\left(\frac{5}{2}\right)} \int_0^x \frac{t}{(x-t)^{-\frac{3}{2}}} dt = \frac{16 x^{\frac{7}{2}}}{105\sqrt{\pi}}.$$

$$y' = \left(- \frac{16\sqrt{x}\left(2 + 2x + x^2\right)}{105\sqrt{\pi}} \right)_{\frac{3}{2}}, \tag{3.5}$$

using definitions of the fractional differintegration,

$$y' = -\frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{d^2}{dx^2} \int_0^x \frac{16\sqrt{t}\left(2 + 2t + t^2\right)}{105\sqrt{\pi}\sqrt{x-t}} dt. \tag{3.6}$$

We obtain the following solution,

$$y' = -\frac{2}{35}(4 + 5x). \tag{3.7}$$

Thus, (3.7) satifies (3.1).

In the same way as the procedure in (y'), *we obtain*

$$y'' = -\frac{2}{35}(4+5x). \tag{3.8}$$

EXAMPLE 3.2. *If we substitute* $\tau = 2$ *and* $\gamma = 1$ *in equation (2.25), then we obtain the following equation,*

$$xy_2 + (1-x)y_1 - 2y = 0. \tag{3.9}$$

Its solution is

$$y = \ell\left(e^x x\right)_1. \tag{3.10}$$

Indeed, after necassary operations in (3.10), we obtain the following solution,

$$\frac{d\ell\left(e^x x\right)}{dx} = \ell\,(1+x)\,e^x = y. \tag{3.11}$$

Differentiating twice (3.11), *we find that*

$$y' = \ell\,(2+x)\,e^x \tag{3.12}$$

$$y'' = \ell\,(3+x)\,e^x. \tag{3.13}$$

Obviously equation (3.11), is solution for equation (3.9)

4. Conclusion

The $N-$fractional calculus operator N^{μ} method is applied to the homogeneous and non-homogeneous a confluent hypergeometric equation. Explicit solution of the equations are obtained.

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